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On Coefficient Bounds of Certain Subfamilies of Close-to-Convex Functions of Complex Order Defined by Sãlãgean Derivatives

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Abstract. Motivated from the recent work of Srivastava et al. (H.M. Srivastava, Qing-Hua Xu, Guang-Ping Wu, Coefficient estimates for certain subclasses of spiral-like functions of complex order, 23 (2010) 763-768), we aim to determine the coefficient estimates for functions in certain subclasses of close-to-convex and related functions of complex order, which are here defined by means of Sãlãgean derivative operator and Cauchy-Euler type non-homogeneous differential equation. Several interesting consequences of our results are also observed.

1. Introduction

Let $\mathcal{A}$ denote the class of function $f(z)$:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let f and g be analytic in E, we say that f is subordinate to g, written as $f(z) \prec g(z)$ if there exists a Schwarz function w, which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ $(z \in E)$, such that $f(z) = g(w(z))$. In particular, when g is univalent, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subseteq g(E)$, see [7]. Also let $S^*(\gamma)$, $C(\gamma)$, $K(\gamma)$ and $Q(\gamma)$ be the subclasses of $\mathcal{A}$ consisting of all functions which are starlike, convex, close-to-convex and quasi convex of complex order γ $(\gamma \neq 0)$ respectively, for details see [1, 9–12]. We note that for $0 < \gamma \leq 1$, these classes coincide with the well known classes of starlike, convex , close-to-convex and quasi convex of order $1 - \gamma$.

Sãlãgean [14] introduced the operator $D^n (n \in N_0)$ which is also called Sãlãgean derivative operator and is defined as:

$D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$,
and, in general,
$D^n f(z) = D(D^{n-1} f(z))$ $(n \in N)$

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or, equivalently,

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \ (n \in N_0 : f \in \mathcal{A}).$$

Let $h : E \longrightarrow \mathbb{C}$ be a convex function such that $h(0) = 1$ and $\mathbb{R}e\, h(z) > 0$ $(z \in E)$. In a recent work Srivastava et al. [22] study the following class of starlike functions,

$$S_h^*(n, \lambda, \gamma) = \left\{ f : f \in \mathcal{A} \text{ and } 1 + \frac{1}{\gamma}\left[\frac{z[(1-\lambda)\, D^n f(z) + \lambda D^{n+1} f(z)]'}{(1-\lambda)\, D^n f(z) + \lambda D^{n+1} f(z)} - 1\right] \in h(E)\ (z \in E)\right\}$$

where $0 \le \lambda \le 1; n \in N_0; \gamma \in \mathbb{C}\backslash\{0\}$. Note that with $h(z) = \frac{1+z}{1-z}$

$$S_h^*(0, 0, \gamma) = S^*(\gamma), \ \ S_h^*(0, 1, \gamma) = C(\gamma).$$

Here we define the following.

Definition 1.
Let $f \in \mathcal{A}$. Then $f \in \mathcal{KQ}_{\langle}(n, \lambda, \gamma)$ if there exists a function $g \in S_h^*(n, \lambda, 1)$ such that

$$1 + \frac{1}{\gamma}\left[\frac{z[(1-\lambda)\, D^n f(z) + \lambda D^{n+1} f(z)]'}{(1-\lambda)\, D^n g(z) + \lambda D^{n+1} g(z)} - 1\right] \in h(E)\ (z \in E) \tag{2}$$

where $(0 \le \lambda \le 1; n \in N_0; \gamma \in \mathbb{C}\backslash\{0\})$.
We note that with $h(z) = \frac{1+z}{1-z}$,

$$\mathcal{KQ}_{\langle}(0, 1, \gamma) = K(\gamma), \ \ \mathcal{KQ}_{\langle}(0, 1, \gamma) = Q(\gamma).$$

Motivated from the recent work of Srivastava et al. [22] the main purpose of our investigation is to derive coefficient estimates of a subfamily $T_h(n, \lambda, \gamma; \mu)$ of $\mathcal{A}$, which consists of functions $f(z)$ in $\mathcal{A}$ satisfying the following Cauchy Euler type non homogenous differential equation

$$z^2 \frac{d^2 w}{dz^2} + 2(1+\mu) z \frac{dw}{dz} + \mu(1+\mu) w = (1+\mu)(2+\mu) h(z), \tag{3}$$

where $w = f(z)$, $h(z) \in \mathcal{KQ}_{\langle}(n, \lambda, \gamma)$, $\mu \in \mathbb{R} - (-\infty, -1]$, for related work see [2–6, 8, 15–27] and the references therein.

2. Preliminary Results

We need the following lemmas, which are essential in our forthcoming results.
Lemma 1 [22]**.** If the function

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in S_h^*(n, \lambda, \gamma),$$

then

$$\left|a_j\right| \le \frac{\prod_{k=0}^{j-2} (k + |h'(0)||\gamma|)}{(j-1)!\, j^n (1 - \lambda + j\lambda)} \ \left(j \in N_0 =: N \backslash \{1\} = \{2, 3, 4 ....\}\right).$$

Lemma 2 [13]**.** Let the function g given by

$$g(z)=\sum_{k=1}^{\infty} b_k z^k,$$

be convex in E. Also let the function f given by

$$f(z)=\sum_{k=1}^{\infty} a_k z^k,$$

be analytic in E. If $f(z) \prec g(z)$ $(z \in E)$, then

$$|a_k| \leq |g_1|$$

3. Coefficient Estimates for Functions in the Class $\mathcal{KQ}_{\langle}(n,\lambda,\gamma)$

Theorem 1.
Let the function f given by (1). If $f \in \mathcal{KQ}_{\langle}(n,\lambda,\gamma)$, then

$$|a_j| \leq \frac{\prod_{k=0}^{j-2}(k+|h'(0)|)}{j^n(1+(j-1)\lambda)\,j!}+\frac{|\gamma|\,|h'(0)|}{j^{n+1}(1+(j-1)\lambda)}\sum_{k=1}^{j-1}\frac{\prod_{k=0}^{j-k-2}(k+h'(0))}{(j-k-1)!}. \tag{4}$$

This result is sharp.
Proof.
Suppose that the functions $F(z)$ and $G(z)$ be defined in terms of the Sãlãgean derivative operator D^n, by

$$\begin{aligned} F(z) &= (1-\lambda)\,D^n f(z)+\lambda D^{n+1} f(z) \\ &= z+\textstyle\sum_{j=2}^{\infty} A_j z^j, \end{aligned} \tag{5}$$

and

$$\begin{aligned} G(z) &= (1-\lambda)\,D^n g(z)+\lambda D^n g(z) \\ &= z+\textstyle\sum_{j=2}^{\infty} B_j z^j, \end{aligned} \tag{6}$$

where

$$A_j = j^n(1+(j-1)\lambda)\,a_j, \text{ and } B_j = j^n((1+(j-1)\lambda)\,b_j.$$

From Definition 1, we have

$$1+\frac{1}{\gamma}\left[\frac{zF'(z)}{G(z)}-1\right] \in h(E)\ (z \in E).$$

Let

$$p(z)=\frac{1}{\gamma}\left[\frac{zF'(z)}{G(z)}-1\right] \in h(E).$$

This implies that

$$zF'(z) = [1+\gamma(p(z)-1)]\,G(z).$$

After some simplification, we get

$$jA_j = B_j+\gamma\sum_{k=1}^{j-1} p_k B_{j-k}$$

$$j\left|A_j\right| \leq \left|B_j\right| + \left|\gamma\right| \sum_{k=1}^{j-1} \left|p_k\right| \left|B_{j-k}\right|.$$

Therefore by using Lemma 1 together with Lemma 2, we have

$$\left|A_j\right| \leq \frac{\prod_{k=0}^{j-2}(k+|h'(0)|)}{j\,(j-1)!} + \frac{\left|\gamma\right| |h'(0)|}{j} \sum_{k=1}^{j-1} \frac{\prod_{k=0}^{j-k-2}(k+|h'(0)|)}{(j-k-1)!}.$$

Hence,

$$\left|a_j\right| \leq \frac{\prod_{k=0}^{j-2}(k+|h'(0)|)}{j^n\,(1+(j-1)\lambda)\,j!} + \frac{\left|\gamma\right| |h'(0)|}{j^{n+1}\,(1+(j-1)\lambda)} \sum_{k=1}^{j-1} \frac{(k+|h'(0)|)}{(j-k-1)!}.$$

This completes the proof of Theorem 1.

We can state the following corollaries:

Corollary 1. Let $h(z) = \frac{1+Az}{1+Bz}$ and $f \in \mathcal{A}$ be given by (1). If $f \in \mathcal{KQ}_\zeta(n,\lambda,\gamma)$, then

$$\left|a_j\right| \leq \frac{\prod_{k=0}^{j-2}(k+(A-B))}{j!\,j^n\,(1+(j-1)\lambda)\,j!} + \frac{\left|\gamma\right| |A-B|}{j^{n+1}\,(1+(j-1)\lambda)} \sum_{k=1}^{j-1} \frac{\prod_{k=0}^{j-k-2}(k+(A-B))}{(j-k-1)!}. \tag{7}$$

The above corollary with $n = 0$ is proved recently in [24].

Corollary 2. Let $h(z) = \frac{1+z}{1-z}$ and $f \in \mathcal{A}$ be given by (1). If $f \in \mathcal{KQ}_\zeta(n,\lambda,\gamma)$, then

$$\left|a_j\right| \leq \frac{1}{j^n\,(1+(j-1)\lambda)} + \frac{\left|\gamma\right|(j-1)}{j^n\,(1+(j-1)\lambda)}. \tag{8}$$

For $\gamma = 1, n = 0$ in (8), we obtain the well known coefficient estimates of close-to-convex (with $\lambda = 0$) and quasi convex (with $\lambda = 1$) mappings respectively.

4. Coefficient Estimates of the Class $T_h(n,\lambda,\gamma;\mu)$

The theorem below is our main coefficient estimates for functions in the class $T_h(n,\lambda,\gamma;\mu)$.
Theorem 2. Let $f \in T_h(n,\lambda,\gamma;\mu)$ and be defined by (1). Then for $n \in N^* = \{2,3,4,\ldots\}$

$$|a_n| \leq \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)}\left[\left|a_j\right| \leq \frac{\prod_{k=0}^{j-2}(k+|h'(0)|)}{j^n\,(1+(j-1)\lambda)\,j!} + \frac{\left|\gamma\right| |h'(0)|}{j^{n+1}\,(1+(j-1)\lambda)} \sum_{k=1}^{j-1} \frac{\prod_{k=0}^{j-k-2}(k+h'(0))}{(j-k-1)!}\right]. \tag{9}$$

Proof. Since $f \in T_h(n,\lambda,\gamma;\mu)$, then there exist $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}_\zeta(n,\lambda,\gamma)$, such that (3) holds true. Thus it follows that

$$a_n = \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)} b_n, \quad n \in N^*,\ \mu \in \mathbb{R} - (-\infty,-1].$$

Hence, by using Theorem 1, we immediately obtain the desired inequality (9).

References

[1] H. S. Al-Amiri and T. S Fernando, On close-to-convex functions of complex order, Internat. J. Math. and Math. Sci. 13(1990), 321-330.
[2] O. Altintas, Ö. Özkan and H. M. Srivastava, Neighbourhoods of a classof analytic functions with negative coefficients, Appl. Math. Lett. 13 (3) (1995), 63-67.
[3] O. Altintas, Ö. Özka and H. M. Srivastava, Majorization by starlike functions of complex order, Complex Variables Theory Appl. 46 (2001), 207-218.
[4] O. Altintas, Ö. Özkan and H. M. Srivastava, Neighbourhoods of a certain family of multivalent functions with negative coefficient, Comput. Math. Appl. 47(2004), 1667-1672.
[5] O. Altintas, H. Irmak, S. Owa and H. M. Srivastava, Coefficient bounds for some familiesof starlike and convex functions of complex order, Appl. Math. Lett. 20(2007), 1218-1222.
[6] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, Filomat 27(2013),1165-1171.
[7] A. W. Goodman, Univalent Functions, Vol. I, Polygonal Publishing House, Washington, New Jersey, 1983.
[8] J. M. Jahangiri, Y. C. Kim and H. M. Srivastava, Construction of acertain class of harmonic close-to-convex functions associated with theAlexander integral transform, Integral Transforms Spec. Funct. 14(2003), 237-242.
[9] O. S. Kown and S. Owa, .On quasi convex functions of complex order, Soochow J. Math. 20 (1994), 241-250.
[10] M. A. Nasr and M. K. Aouf, Radius of convexity for the class of starlike functions of complex order, Bull. Fac. Sci. Assiut Univ. Sect. A 12(1983), 153-159.
[11] K. I. Noor, Quasi-convex functions of complex order, PanAmer. Math. J. 3(2)(1993), 81-90.
[12] S. Owa, M. Nunokawa, H. Saitoh and H. M. Srivastava, Close-to-convexity,starlikeness, and convexity of certain analytic functions, Appl. Math. Lett.15(1)(2002), 63-69.
[13] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc. (Ser. 1) 48 (1943), 48-82.
[14] G. S Sālāgean, Subclass of univalent functions, in: Complex Analysis: Fifth Romanian-Finisg Seminar, Part 1 (Bucharest, 1981), in: Lecture Notes in Mathematics, vol. 1013, Springer-Verlag, Berlin, Heidelberg and New York, 1983, pp. 362-372.
[15] H. M. Srivastava, Subordinations for multivalent analytic functions associated with the Dzoik-Srivastava operator, Integral Transforms Spec. Funct. 20(2009), 581-606.
[16] H. M. Srivastava and S. Owa, Certain classes of analytic functions with varying arguments, J. Math. Appl. 136(1988), 217-228.
[17] H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.
[18] H. M. Srivastava, O. Altıntaş and S. K. Serenbay, Coefficient bounds for certain subclasses of starlike functions of complex order, Appl. Math. Lett. 24(2011), 1359-1363.
[19] H. M. Srivastava, S.-H. Li and H. Tang, Certain classes of k-uniformlyclose-to-convex functions and other related functions defined by using the Dziok-Srivastava operator, Bull. Math. Anal. Appl. 1 (3) (2009), 49-63(electronic).
[20] H. M. Srivastava, A. K. Mishra and M. K. Das, The Fekete-Szegö problem for a subclass of close-to-convex functions, Complex Variables Theory Appl. 44(2001), 145-163.
[21] H. M. Srivastava, D. Raducanu and G. S. Salagean, A new class of generalized close-to-starlike functions defined by the Srivastava-Attiya operator, Acta Math. Sinica (English Ser.) 29 (2013), 833-840.
[22] H. M. Srivastava, Q.-H. Xu and G.-P. Wu, Coefficient estimates for certain subclasses of spiral-like functions of complex order, Appl. Math. Lett. 23(2010), 763–768.
[23] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27(2013), 831–842.
[24] W. Ul-Haq, A. Nazneen and N.Rehman, Coefficient estimates for certain subfamilies of close-to-convex functions of complex order, Filomat, in press.
[25] W. Ul-Haq, A. Nazneen, M. Arif and N.Rehman, Coefficient bounds for certain subclasses of close-to-convex functions of Janowski type, J. Comp. Anal. Appl, 16(1)(2014), 133-138.
[26] Q.-H. Xu, H. M. Srivastava and Z. Li, A certain subclass of analytic and close-to-convex functions, Appl. Math. Lett. 24 (2011), 396-401.
[27] Q.-H. Xu, C.-B. Lv, N.-C. Lou and H. M. Srivastava, Coefficient estimates for the inverses of a certain general class of spirallike functions, Filomat 27(2013), 1351–1356.